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A Geometric Characterization of Dowling Lattices

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The Dowling lattices based on finite groups are generalizations of the partition lattices. They form one of the two classes of modularly complemented geometric lattices. In this paper, we present a characterization of the Dowling lattices in terms of the existence of special bases and their weight functions. We show that the hypotheses of this characterization allow us to appeal to a powerful result of Kahn and Kung. A characterization of the partition lattices, Π_n, follows as a corollary of our main result.

0. INTRODUCTION

The Dowling lattices are generalizations of the partition lattices, Π_n. We briefly sketch the two steps Dowling used to describe this generalization. We refer the reader to Dowling [1] for a complete treatment of these lattices, and to Welsh [5] for matroid terminology, which we will find convenient to use.

First we consider partial partitions. A *partial partition* of an n-set $N = \{1, ..., n\}$ is a collection of non-empty, disjoint subsets of N. Thus each partial partition of N is a partition of some subset of N. The partial partitions are ordered by refinement: for π, ρ partial partitions of N, $\pi \leq \rho$ (or π is a refinement of ρ) if and only if each block of ρ is a union of blocks of π. With this ordering, the partial partitions of N form a geometric lattice, denoted Q_n. The partition of N into singleton sets is a refinement of every partial partition of N, and thus it is the least element of Q_n. The greatest element of Q_n is the empty partial partition — i.e., the partial partition with no blocks. Covers of a partial partition π are formed in two ways: a I-cover is formed by deleting some block of π, whereas a II-cover is formed by "replacing" two blocks of π by their union and leaving the other blocks

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unchanged. In particular, there are two types of atoms: a I-atom consists of $n-1$ blocks containing one element each, and a II-atom consists of $n-2$ one-element blocks and a single two-element block. It is easy to see that there are n I-atoms, and that these form a basis for Q_n. From our description of covers, it follows that the rank of the partial partition π is $n-|\pi|$, where $|\pi|$ denotes the number of blocks of π. We can associate a partition π' of $N \cup \{0\}$ with each partial partition π of N by collecting the "unused" elements of π into a block containing 0 and leaving the blocks of π unchanged. In this way, it becomes clear that Q_n is isomorphic to Π_{n+1}.

We next consider partial G-partitions. Let G be a finite group. A *G-labeling* of $A \subseteq N$ is a function a: $A \to G$. A *partial G-partition* of N is a collection of G-labelings

$$a_1 : A_1 \to G, ..., a_k : A_k \to G$$

for which the sets $A_1, ..., A_k$ form a partial partition of N. The partial G-partitions are equated in a "projective" manner: $\{a_1, ..., a_k\}$ is equivalent to $\{b_1, ..., b_k\}$ if each b_i is a left G-multiple of some a_j. An ordering on the equivalence classes of partial G-partitions is defined as follows. The equivalence class of partial G-partitions containing $a_1: A_1 \to G, ..., a_k: A_k \to G$ is less than or equal to that containing $b_1: B_1 \to G, ..., b_h: B_h \to G$ if

(1) $\{A_1, ..., A_k\} \leq \{B_1, ..., B_h\}$ as partial partitions, and

(2) whenever $A_i \subseteq B_j$, then the restriction of b_j to A_i is a left G-multiple of a_i.

With this ordering, the equivalence classes of partial G-partitions of N form a geometric lattice. This lattice is known as the rank n Dowling lattice based on G, and is denoted by $Q_n(G)$. In particular, the rank n Dowling lattice over a one-element group is the partition lattice Π_{n+1}: obviously the labelings in this case have little significance. Our observations about covers and the rank function of Q_n carry over to $Q_n(G)$ with only minor modifications to account for the labelings. Again in this setting, there are two types of atoms. A I-atom is represented by $n-1$ labelings defined on distinct one-element subsets of N. A II-atom is represented by $n-2$ labelings defined on distinct one-element subsets of N together with a labeling defined on the remaining two elements of N. Again, there are n I-atoms, and they form a basis for $Q_n(G)$.

Dowling lattices are important for two main reasons. First, they give rise to a large class of non-representable matroids: Dowling has shown that $Q_n(G)$ is representable if and only if G is a subgroup of the group of units of a finite field. Second, Dowling lattices are modularly complemented. This aspect plays an important role in our work.

DEFINITION. A geometric lattice is *modularly complemented* if each element of the lattice has a complement which is modular.

K. Bogart introduced modularly complemented geometric lattices in conversations with J. R. Stonesifer that led to [4]. These lattices lie between the highly structured modular geometric lattices (a product of projective spaces and a Boolean algebra) and arbitrary geometric lattices. It is well known (see [3]) that a rank n geometric lattice is modularly complemented if and only if for each point (element of rank 1) there is a modular copoint (element of rank $n-1$) not containing the point. We refer to this second formulation by saying that the lattice is *atom-modularly complemented.*

The Classification Theorem of [2] shows that there are relatively few modularly complemented geometric lattices:

THEOREM (Kahn and Kung). *Any connected modularly complemented geometric lattice of (finite) rank at least four is either*

(a) *a Dowling lattice over some group G, or*

(b) *the lattice of flats of a subgeometry of the rank n projective geometry, $PG(n-1, \tilde{k})$, containing all the internal points relative to some fixed basis of $PG(n-1, \tilde{k})$.*

In part (b) of this theorem, $\tilde{k}$ is a finite (skew) field. The notion of internal points, refered to in the theorem, is connected with the concept of weight, which is fundamental in this paper.

DEFINITION. Let B be a basis of the geometric lattice L. An atom p of L has *weight k with respect to B* if k is the least integer such that p is in a flat spanned by some k elements of B. The weight of p with respect to B is denoted $\mathrm{wt}_B(p)$. Atoms of weight less than n are called *internal points with respect to B.*

An atom p is in the basis B if and only if $\mathrm{wt}_B(p)=1$. Assume for the moment that p is not in B. Using a bit of matroid theory, let $B_p \subseteq B$ be such that $B_p \cup \{p\}$ is the fundamental circuit of p in the basis B. It is well known that B_p is the unique subset of B of size $\mathrm{wt}_B(p)$ having p in its closure, and that any subset of B having p in its closure contains B_p.

DEFINITION. Let $B=\{b_1, ..., b_n\}$ be a basis of the geometric lattice L. We say that an atom p *depends on b_i with respect to B* if $b_i \in B_p$. Otherwise, p is *independent of b_i.*

Recall that to coordinatize $(n-1)$-dimensional Euclidean space, thought of as rank n affine space, we choose n affinely independent points (a basis in matroid terminology), choose one to be a zero point, and use the lines joining that point to the other basis points as our coordinate lines.

DEFINITION. Given a basis $B=\{b_1, ..., b_n\}$ for a geometric lattice, we define the *coordinate lines of B through b_i* to be the lines $b_i \vee b_j$ for $j=1, ..., i-1, i+1, ..., n$.

1. A CHARACTERIZATION OF DOWLING LATTICES

We now state our characterization of Dowling lattices. We will prove the characterization through a series of propositions.

THEOREM 1. *A finite geometric lattice, L, of rank greater than three is a Dowling lattice $Q_n(G)$, for some group G, if and only if there is a basis $B=\{b_1, ..., b_n\}$ of L satisfying*:

(1) *all points have weight* 1 *or* 2 *with respect to B,*

(2) *the coordinate line $b_i \vee b_j$ contains at least three points for all i, j, and*

(3) *if p and q are non-basis points on distinct coordinate lines through b_i, then the line $p \vee q$ contains at least three points.*

PROPOSITION 1. *The Dowling lattice $Q_n(G)$ has a basis satisfying conditions* (1)–(3).

Proof. Recall the I-atoms in $Q_n(G)$ from Section 0. It is straightforward to check, using appropriate definitions and results from [1], that these atoms form a basis satisfying conditions (1)–(3). ▮

Several comments are in order about this basis for $Q_n(G)$. It is easy to show that, if G is not the trivial group, then the I-atoms form the unique basis satisfying (1). Also, the conclusion of (3) fails whenever exactly one of p and q is a basis point. Similar statements will be made in Section 2 about the partition lattice, which arises when G is the trivial group. The I-atoms form the "distinguished" basis refered to in Kahn and Kung [2].

LEMMA 1. *If L is a geometric lattice satisfying condition* (1) *of Theorem* 1, *then each non-basis point p lies on a unique coordinate line.*

Proof. From (1), p lies on some coordinate line. From our remarks after the definition of weight, this coordinate line is unique. ▮

LEMMA 2. *Assume L is a geometric lattice which has a basis $B=\{b_1, ..., b_n\}$ such that all points have weight* 1 *or* 2 *with respect to B.*

Assume that p_j is a non-basis point on the line $b_i \vee b_j$ and that p_k is a non-basis point on the line $b_i \vee b_k$, with $j \neq k$. Let q be a point on the line $p_j \vee p_k$, other than p_j or p_k. Then $q \notin B$ and q lies on the coordinate line $b_j \vee b_k$. Furthermore, $p_j \vee p_k$ is a three point line.

Proof. Since $q \leq p_j \vee p_k \leq b_i \vee b_j \vee b_k$, and q is on some coordinate line, q is on one of the coordinate lines $b_i \vee b_j$, $b_i \vee b_k$, or $b_j \vee b_k$. Assume, for the moment, that q is on $b_i \vee b_j$. Then

$$p_j \vee p_k = p_j \vee q = b_i \vee b_j.$$

Therefore p_k is on the distinct coordinate lines $b_i \vee b_k$ and $b_i \vee b_j$. Since p_k is on a unique coordinate line by Lemma 1, q is not on $b_i \vee b_j$. Similarly, q is not on the coordinate line $b_i \vee b_k$. Therefore q is on $b_j \vee b_k$.

If $p_j \vee p_k$ contained four or more points, say $p_j, p_k, q, q', \ldots$, then q and q' are both on the coordinate line $b_j \vee b_k$. Thus,

$$p_j \vee p_k = q \vee q' = b_j \vee b_k.$$

But this would imply that p_j is on the distinct coordinate lines $b_i \vee b_j$ and $b_j \vee b_k$. Hence $p_j \vee p_k$ is a three point line. ▮

It is worth noting that we can obtain the following corollary by a geometric argument similar to the usual arguments of projective geometry.

COROLLARY. *Assume that L is a geometric lattice and $B = \{b_1, \ldots, b_n\}$ is a basis of L satisfying conditions* (1)–(3) *of Theorem* 1. *Then all coordinate lines $b_i \vee b_j$ contain the same number of points.*

Proof. It suffices to show that for any three distinct integers i, j, and k, we have $|b_i \vee b_k| \leq |b_j \vee b_k|$. Fix a point p_j on $b_i \vee b_j$ with p_j not in B. By Lemma 2, for each $p \in b_i \vee b_k$, with p not in B, we get a unique point q on the line $b_j \vee b_k$, again with q not in B. It suffices to show that this map is injective. Assume that p corresponds to q in this way, and that p' corresponds to q'. Thus $\{p, p_j, q\}$ and $\{p', p_j, q'\}$ are flats by the last assertion of Lemma 2. Hence if $q = q'$, it follows that $p = p'$, proving that our map is injective. ▮

Returning to the converse of Theorem 1, our tactic is to construct a special collection of modular copoints which we will use to show that the lattice is modularly complemented. We can then appeal to Kahn and Kung's Classification Theorem to complete the argument.

PROPOSITION 2. *If a geometric lattice L has a basis satisfying conditions* (1)–(3) *of Theorem* 1, *then L is modularly complemented.*

Proof. Let $b_1, ..., b_n$ be as in Theorem 1, and let $\hat{b}_i = b_1 \vee \cdots \vee b_{i-1} \vee b_{i+1} \vee \cdots \vee b_n$. We want to show that $\hat{b}_i$ is modular. Note that since $\hat{b}_i$ is a copoint, it suffices to show that if $y \not\leq \hat{b}_i$, then $y \wedge \hat{b}_i$ is covered by y. So assume $y \not\leq \hat{b}_i$, and let $Q = \{q_1, ..., q_h\}$ be a basis for y. We may assume that $h \geqslant 2$, and that Q has been chosen so that $|Q \cap B|$ is maximum. We will produce a basis of $h-1$ elements for $y \wedge \hat{b}_i$. Producing this basis is easily handled in two cases.

Case 1. b_i is in Q. Assume $q \in Q - B$. We claim that q is independent of b_i. Once we have shown this, it follows that only one element of Q, namely b_i, fails to be less than or equal to $y \wedge \hat{b}_i$. The remaining elements of Q give us a basis of $y \wedge \hat{b}_i$ having $h-1$ elements.

Now, if $q \leq b_i \vee b_j$, then $b_i \vee q = b_i \vee b_j$. Thus b_j could replace q, producing a basis for y with more elements from B, contrary to our assumption that Q has been chosen with $|Q \cap B|$ maximum.

Case 2. b_i is not in Q. Assume $q_s, q_t \in Q - B$ with $q_s \leq b_i \vee b_j$ and $q_t \leq b_i \vee b_k$. We claim that we can replace one of q_s and q_t by another atom p, again giving a basis of y, with p independent of b_i. Once we have shown this, we can perform this replacement sufficiently often to produce a basis for y having $h-1$ elements less than or equal to $\hat{b}_i$.

If $b_j = b_k$, where $q_s \leq b_i \vee b_j$ and $q_t \leq b_i \vee b_k$, then $q_s \vee q_t = b_i \vee b_j = b_i \vee b_k$. This is impossible by the way we chose of Q. Hence, the coordinate lines $b_i \vee b_j$ and $b_i \vee b_k$ are distinct. Thus, by hypothesis (3), the line $q_s \vee q_t$ contains some third point p. From Lemma 2, p is independent of b_i. Because $q_s \vee q_t = q_s \vee p = p \vee q_t$, it is clear that p can replace one of q_s and q_t.

These two cases show that $\hat{b}_i$ is modular. Since $b_i \leq \bigwedge_{j<i} \hat{b}_j$ (this meet is the greatest element, 1, if $i=1$) and $b_i \not\leq \hat{b}_i$, it follows that $\bigwedge_{j<i} \hat{b}_j \not\leq \hat{b}_i$. Inductively, we obtain $r(\bigwedge_{j\leqslant i} \hat{b}_j) = n-i$. In particular, $r(\bigwedge_{j=1}^{n} \hat{b}_j) = 0$. From this it is clear that L is atom-modularly complemented, and hence L is modularly complemented. ∎

Now we may complete the proof of Theorem 1.

PROPOSITION 3. *If a geometric lattice L of rank greater than three has a basis satisfying conditions* (1)–(3) *of Theorem* 1, *then L is a Dowling lattice* $Q_n(G)$ *for some group G.*

Proof. From condition (2), L is simple (i.e., indecomposable—connected, in matroid terminology). For if $L = L_1 \times L_2$, then any basis of L must have atoms of the forms $x = (a, 0_2)$ and $y = (0_1, b)$ for a an atom of L_1, b an atom of L_2. But clearly then $x \vee y$ would be a two point line.

From Proposition 2, L is modularly complemented. Thus the Classification Theorem applies. Among lattices in the Classification Theorem, only the Dowling lattices satisfy (1). Hence L is a Dowling lattice. ∎

2. A Characterization of Partition Lattices

Theorem 2. *A rank n geometric lattice L is the partition lattice Π_{n+1} if and only if there is a basis $B = \{b_1, ..., b_n\}$ of L satisfying:*

(1) *all points have weight 1 or 2 with respect to B,*

(2) *the coordinate line $b_i \vee b_j$ contains at least three points for all i, j, and at least one $b_i \vee b_j$ is a three point line, and*

(3) *if p and q are non-basis points on distinct coordinate lines through b_i, then the line $p \vee q$ contains at least three points.*

Proof. It is straightforward to check that the n partitions whose only non-trivial cells are $\{1, 2\}$, $\{1, 3\}$, ..., $\{1, n+1\}$ respectively, serve as such a basis for Π_{n+1}.

Turning to the converse, the corollary of Lemma 2 implies in (2) that each $b_i \vee b_j$ is a three point line. Lemma 2 implies in (3) that $p \vee q$ is a three point line.

For $n = 2$, the converse is trivial. For $n = 3$, we use the notation $b_i \vee b_j = \{b_i, b_j, p_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$. By (3), $\{p_1, p_2, p_3\}$ is a line, and clearly $b_i \vee p_i$ is always a two point line. This information allows us to draw the lattice very easily. When we do so, the isomorphism with Π_4 is evident.

For $n > 3$, Theorem 1 implies that L is a Dowling lattice. Among all rank n Dowling lattices, the partition lattice Π_{n+1} has the fewest points, namely $\binom{n+1}{2}$. Note that (1), (2), and Lemma 1, imply that L has $n + \binom{n}{2} = \binom{n+1}{2}$ points. Hence, L must be Π_{n+1}. ∎

In the case of the partition lattice we see that, up to permutations of $\{1, ..., n+1\}$, the basis we described at the beginning of the proof is the only basis of Π_{n+1} satisfying conditions (1)–(3). Again, the conclusion of (3) fails whenever exactly one of p and q is a basis point.

We would like to point out a fact that plays an important role in many arguments about modularly complemented geometric lattices, and yet seems never to have been explicitly stated. A rank n geometric lattice is modularly complemented if and only if it contains a rank n subgeometry of modular elements isomorphic to a Boolean algebra.

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